UNIT 8 : CONTINUOUS FUNCTIONS

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8.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to

- define and learn about continuity of a function
- know examples of continuous functions
- know about the types of discontinuity
- know examples of discontinuous functions
- learn about the properties of continuous functions
- define and know about uniform continuity of a function.

8.2 INTRODUCTION

In the previous unit, we studied about the limit of functions. In this unit, we study about continuous functions which comprise of an important special class of functions in real analysis. The word “continuous” means “without interruption”. In case of functions, the word “continuity” holds the same meaning. Those function whose graph can be drawn without interruption (without holes or breaks) can be referred to as continuous function. In section, we discuss about the properties of continuous functions

which make it so important. We introduce the concept of uniform continuity in section.

8.3 DEFINITION AND EXAMPLE OF CONTINUOUS FUNCTION

Definition 1: Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be continuous at *c*, if the following conditions are satisfied:

1. *f*(*c*) is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

If *f* fails to be continuous at *c*, then *f* is said to be discontinuous at *c*.

Definition 2: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $B \subseteq A$, then the function *f* is said to be continuous on the set *B* if *f* is continuous at every point of *B*.

Example : Prove that the constant function $f(x) := a$ is continuous on $\mathbb{R}$.

We first prove the continuity of the constant function at a point $c \in \mathbb{R}$. We know that $f(c) = a$ is defined. Also, $\lim_{x \to c} a = a = f(c)$ for any constant function. So, from the definition 1 of continuity, *f* is continuous at $x = c$ for any $c \in \mathbb{R}$. Henceforth, *f* is continuous on $\mathbb{R}$.

Example : Prove that the identity function $\lim_{x \to c} a = a = f(c)$ is continuous at $\mathbb{R}$.

We first prove the continuity of the identity function at a point $c \in \mathbb{R}$. We know that $f(c) = c$ is defined. Also, $\lim_{x \to c} x = c = f(c)$. So, from the definition 1 of continuity, *f* is continuous at *x* = *c* for any $c \in \mathbb{R}$. Henceforth, *f* is continuous on $\mathbb{R}$.

Example : Prove that the modulus function $f(x) := |x|$ is a continuous function on $\mathbb{R}$.

We can write |*x*| as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } < 0 \end{cases}$$

It is clear that $f(x) := |x|$ is continuous for all *x* > 0 and *x* < 0. So, to

prove the continuity on $\mathbb{R}$, we only need to prove the continuity of $|x|$ at x = 0. Since $f(0)=|0|=0$ is defined, so to prove the continuity at x = 0, we need to prove that $\lim_{x\to 0}|x|=f(0)=0.$ As studied in the previous chapter, we know that $\lim_{x\to 0}+|x|=\lim_{x\to 0}+x=0$ and $\lim_{x\to 0}-|x|=\lim_{x\to 0}-(-x)=0.$ Since $\lim_{x\to 0}|x|=\lim_{x\to 0}-|x|=0,$ henceforth $\lim_{x\to 0}|x|=0=f(0).$ Thus modulus function is continuous on $\mathbb{R}$.

In other words, we can define continuous function as follows :

Definition 3 : Let $A\subseteq\mathbb{R}$ and $A\to\mathbb{R}$ be a function with domain A and co-domain $\mathbb{R}$. For any $c\in A,$ the function f is said to be continuous at c, if, given any number $\in>0,$ there exist a number $\delta>0$ such that for any $x\in A, |f(x)-f(c)|<\in$ whenever $|x-c|<\delta.$

Note: The $\in-\delta$ definition of continuity is also known as the Cauchy's definition.

Example: Prove using $\in-\delta$ definition of continuity, that the function $f(x)$: = x is continuous at a point x = 2.

We need to show that for any $\in>0,$ there exists $\delta>0,$ such that $|f(x)-f(2)|<\in$ whenever $|x-2|<\delta.$

Here $f(2)$ = 2. So $|f(x)-f(2)|=|x-2|$.

Now, for $\delta=\in,$ it is clear that $|f(x)-f(2)|<\in|$ whenever $|x-2|<\delta.$

Example: Using $\in-\delta$ definition of continuity, prove that $f(x)=4x-5$ is a continuous function of x = 3.

We need to show that for any $\in>0,$ there exists $\delta>0,$ such that $|f(x)-f(3)|<\in$ whenever $|x-3|<\delta.$

Here, $f(3)=7.$ So $|f(x)-f(3)|=|4x-5-7|=4x-12|=4|x-3|.$

So for $\in>0, |f(x)-f(3)|<\in\Rightarrow 4|x-3|<\in\Rightarrow|x-3|<\frac{\in}{4}$

Now, taking $\delta=\frac{\in}{4},$ it is clear that $|f(x)-f(2)|<\in$ whenver $|x-2|<\delta.$

Example: Using $\in-\delta$ definition of continuity, prove that $f(x)=x^2$ is a continuous function at $x=3.$

We need to show that for any $\in>0,$ there exists $\delta>0,$ such that $|f(x)-f(3)|<\in$ whenever $|x-3|<\delta.$

Here, $f(3)$ = 9. So $|f(x)-f(3)=|x^2-9|=|x^2-3^2|=|x-3||x+3|$

So for $\in>0, |f(x)-f(3)|<\in\Rightarrow|x-3||x+3|<\in$

Here $|x+3|$ need a bound so that we can proceed further.

Let us take $\delta = 1$. Then $|x+3|<1 \Rightarrow 2<x<4$.

Now, taking $\delta = \frac{\in}{4}$, it is clear that $|f(x)-f(2)|<\in$ whenever $|x-2|<\delta$.



CHECK YOUR PROGRESS 1

Q 1: Prove that $f(x)=1/x$ is continuous on $A := \{x \in \mathbb{R}: x > 0\}$

Q 2: Determine a condition on $|x-1|$ that will assure that $|x^2-1|<\frac{1}{2}$

Q 3: Using $\in -\delta$ definition of continuity, prove that $f(x)=x^3$ is continuous at $x = 1$.

Q 4: Let $k>0$ and let $f: \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x)-f(y)| \leq k|x-y|$ for all $x, y \in \mathbb{R}$. Show th f is continuous at every point $c \in \mathbb{R}$.

Q 5: Prove that the function $f(x) := \frac{x^3-4}{x^2+1}$ is continuous at $x = \frac{4}{5}$.

8.4 TYPES OF DISCONTINUITY WITH EXAMPLES

In the previous section, we defined the continuity of a function at a point c. We see in Definition 1 that if one or more conditions of this definition fails to hold, then the function is discontinuous at point $x = c$. Based on the failure of one or more out of the three conditions in definition..... types of discontinuity that occurs are as follows :

1. **jump discontinuity** occurs when the right-hand limit and left-hand limit both exist but are not equal. Hence $\lim_{x \to x} f(x)$ does not exist, violating the second condition.
2. **infinite discontinuity** occurs when one-sided limits (left-hand limit and right hand limit) of the function is infinite. Thus $\lim_{x \to x} f(x)$ does not exist, violating the second condition.
3. **removable discontinuity** occurs when $\lim_{x \to x} f(x)$ exists but is not equal to $f(c)$. This violates the third condition of definition on continuity.

Note : Discontinuity occurs when the function is not defined at c. Based on the types of discontinuities we present few examples below.

Example : Show that the function $f(x)=\frac{x^2-9}{x-3}$ has discontinuity at x = 3. The given function $f(x)=\frac{x^2-9}{x-3}$ is not defined at x = 3. The condition 1 of the definition of continuity is not satisfied and henceforth the function is discontinuous at x = 3.

Example : Show that the function $g(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$ has a removable discontinuity at x = 2.

Here we see that $\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=4$

But $f(2)=3$. The $f(x)$ has removable discontinuity at $x=2$.

Example : Verify the continuity of the function $g(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$ at x = 2.

Here

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=4=f(2)$$

Hence the function is continuous at x = 2.

Note : The removable discontinuity in any function can be removed by..... as can be seen in the above examples.

Example : Show that the function $g(x)=\begin{cases}3, & x\geq 3\\ 2, & x<3\end{cases}$ has a jump discontinuity at x = 3.

Here $\lim_{x\to 3+} f(x)=\lim_{x\to 3+}3=3$ but $\lim_{x\to 3-} f(x)=\lim_{x\to 3-}2=2$

Hence $\lim_{x\to 3-} f(x)$ does not exists. This violated the condition 2 of definition 1. So, the function has jump discontinuity at x = 3.

Example : Show that the function $g(x)=\frac{x-2}{x^2-5x+6}$ has a infinite discontinuity at x = 3

Here $\lim_{x\to 3}\frac{x-2}{x^2-5x+6}=\lim_{x\to 3}\frac{x-2}{(x-2)(x-3)}=\lim_{x\to 3}\frac{1}{x-3}$

It is clear that both left-hand limit and right-hand limit of $f(x)$

approaches $\pm\infty$ as x approaches 3. Hence, there exist infinite discontinuity at $x = 3$.



CHECK YOUR PROGRESS 2

Q 1: Verify the continuity of the signum function at $x = 0$.

Q 2: Determine the points of continuity of the greatest integer function $f(x) := [x]$

Q 3: Show that $f(x) = 1/x$ is not continuous at $x = 0$.

Q 4: Show that the function $f(x) := x\sin(1/x),\ x \neq 0$ is not continuous at $x = 0$ but $F(x) = \begin{cases} 0, & x = 0 \\ x\sin(1/x), & x \neq 0 \end{cases}$ is continuous at $x = 0$.

8.5 ALGEBRA OF CONTINUOUS FUNCTION

In this section, we look at the algebra of continuous functions for the four arithmetic operations viz. addition, subtraction, multiplication and division.

Theorem : Let $A \subseteq \mathbb{R}$ and f and g be functions from A to $\mathbb{R}$ which are continuous at $x = c$. Then

(a) $f + g$ is continuous at c

(b) $f - g$ is continuous at c

(c) fg is continuous at c

(d) bf is continuous at c

(e) bg is continuous at c

(f) f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.

Proof : (a) Since f and g are continuous at c, so by definition of continuity we have

$$\lim_{x\to c} f(x) = f(c) \text{ and } \lim_{x\to c} g(x) = g(c)$$

But we know from properties of limit that

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c)$$

Therefore $f + g$ is continuous by definition 1 of continuity.

(b) Similar as above, we have from properties of limit

$$\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = f(c) - g(c)$$

Therefore $f-g$ is continuous by definition 1 of continuity.

(c) In similar fashion, from the properties of limit we have

$$\lim_{x\to c}(fg(x)) = \lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right) = f(c)g(c) = fg(c)$$

Therefore *fg* is continuous by definition 1 of continuity.

(d) Let $g(c)\neq 0.$ Then by property of limit, we have

$$\lim_{x\to c}\frac{f}{g}(x) = \lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c\, g(x)}} = \frac{f(c)}{g(c)} = \frac{f}{g}(c)$$

Hence, *f/g* is continuous at *c* for $g(c)\neq 0.$

For the case $g(c)=0, \dfrac{f(c)}{g(c)}$ is undefined and therefore by first condition of definition 1 of continuity, the function *f/g* is discontinuous at *c*.

Example : Prove that $\lim_{x\to c} x^n = c^n$ for any positive integer *n*.

From the example on the previous section, we know that $\lim_{x\to c} x = c$.

Hence, from the algebra of continuous function, we have

$\lim_{x\to c} x^2 = (\lim_{x\to c} x)(\lim_{x\to c} x) = c^2$

Continuing as above, we get

$\lim_{x\to c} x^3 = (\lim_{x\to c} x^2)(\lim_{x\to c} x) = c^3$

Henceforth, $\lim_{x\to c} x^n = (\lim_{x\to c} x^{n-1})(\lim_{x\to c} x) = c^n$

Example : Prove that polynomial functions are continuous on $\mathbb{R}$.

A polynomial function is of the form $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ for all $\in \mathbb{R}$, where $a_0, a_1, ...a_n$ are constants.

For any point *x* = *c*, *p*(*c*) is defined and given as $p(c) = a_0 + a_1c + a_2c^2 + ... + a_nc^n$

From the algebra of continuous functions, we have

$$\begin{aligned}\lim_{x\to c} p(x) &= \lim_{x\to c}\left(a_0 + a_1x + a_2x^2 + ... + a_nx^n\right)\\ &= \lim_{x\to c} a_0 + \lim_{x\to c} a_1x + \lim_{x\to c} a_2x^2 + ... + \lim_{x\to c} a_nx^n\\ &= a_0 + a_1c + a_2c^2 + ... + a_nc^n\\ &= p(c)\end{aligned}$$

Hence polynomial function are continuous on $\mathbb{R}$

Next theorem shows that the composition of two continuous function is continuous.

Theorem : For *A, B* $\subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous functions at $c \in A$ and for $f(A) \subseteq B$, let $g : B \to \mathbb{R}$ be a continuous functions $b = f(c) \in B$. Then the composition $gof : A \to \mathbb{R}$ is continuous at *c*.

Proof : Since *g* is continuous function at *b* = *f*(*c*), so for $y = f(x) \in B$, we have

$$|g(y) - g(b)| < \in, \text{ whenever } |y - b| < \delta_0$$

or we can write

$$|g(f(x)) - g(f(c))| < \in, \text{ whenever } |f(x) - f(c)| < \delta_0$$

But for any $\delta_0 > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \delta_0 \text{ whenever } |x - c| < \delta$$

Henceforth, for any $\in > 0$, there exists $\delta > 0$, such that

$$|g(f(x)) - g(f(c))| < \in, \text{ whenever } |x - c| < \delta$$

Hence the result.

Theorem : For $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous functions on *A* and for $f(A) \subseteq B$, let $g : B \to \mathbb{R}$ be a continuous functions on B. Then the composition $A \to \mathbb{R}$ is continuous on *A*.

Proof : The theorem follows from the preceding result.

8.6 PROPERTIES OF CONTINUOUS FUNCTION

At first we define the Sequential criterion of continuity and later move to discuss about the functions continuous on intervals.

Sequential criterion for continuity : A function $f : A \to \mathbb{R}$ is continuous at *x* = *c* if and only if for every sequence (x_n) in *A* that converges to *c*, the sequence $(f(x_n))$ converges to *f*(*c*)

Proof : Let $f \to f(c)$ as $x \to c$, i.e. for any $\in > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \in \text{ whenever } |x - c) < \delta$$

Again, let (x_n) be a sequence converging to *c*. So there exists a natural number *m* such that

$$|x_n - c| < \delta, \text{ for } n \geq m$$

Hence $|f(x) - f(c)| < \in$, for $n \geq m$

i.e. the sequence $(f(x_n))$ converges to *f*(*c*).

Conversely, let for every sequence (x_n) in *A* that converges to *c*, the sequence $(f(x_n))$ converges to *f*(*c*).

Suppose that the function *f* has no limit i.e. there exists at least one $\in_0 > 0$ for which δ does not exist. This means there is $x = x^{(\delta)}$ such that

$$|x - c| < \delta, \text{ implies } \left|f\left(f^{(\delta)}\right) - f(c)\right| \geq \in$$

Thus for successive values of $\in_0$, there is value x_k such that

$|x - c| <$ and $|x - c| <$. This implies that $x_k (x_k \neq c)$ tend to *c* but $(f(x_k))$ does not tend to $f(c)$ which is a contradiction. Hence our supposition that limit does not exist is disproved.

Functions that are continuous on intervals have many important properties which are not possessed by general continuous functions. We establish some of the important results here in this section. We start by the general definition of a bounded function.

Definition : A function $f : A \to \mathbb{R}$ is said to be bounded on *A* if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

Note: Continuous functions need not be bounded.

Example: The function $f : (0, \infty) \to \mathbb{R}$ such that $f(x) := 1/x$ is continuous on $(0, \infty)$. But *f* is not bounded on $(0, \infty)$ because for any *M* > 0, there exist point $x_m = 1/(M+1)$ in $(0, \infty)$ such that $f(x_m) = \dfrac{1}{x_m} = M + 1 > M$.

We now state an important theorem based on the boundedness of a continuous function.

Boundedness Theorem : Let $f : I \to \mathbb{R}$ be continuous on closed bounded interval $I := [a, b]$. Then *f* is bounded on *I*.

Proof : We show contra-positively, that if the function is not bounded on *I*, then it fails to be continuous at some point in the closed interval.

Let if possible *f* is not bounded above.

So for any $n \in \mathbb{N}$, there exist a number $x_n \in I$ such that $|f(x_n)| > n$.

Now (x_n) being a sequence in the closed bounded interval *I* is bounded and has at least one limit point, say ϕ where $\phi \in I$.

Further, since ϕ is a limit point of the sequence (x_n), therefore, there exists a sub-sequence $(x_{n_r})$of (x_n) that converges to ϕ as $r \to \infty$.

Since *I* is closed and the elements of (x_{n_r}) belong to *I*, so $\phi \in I$.

Then, *f* is continuous at ϕ. So $(f(x_n))$ converges to $f(\phi)$.

Thus the convergent sequence $f(x_n)$ must be bounded.

But $\left|f\left(x_{n_k}\right)\right| > n_k \geq r$ for all $r \in \mathbb{N}$, which is a contradiction.

Thus the result follows.

Theorem : Let $f : I \to \mathbb{R}$ be continuous function on $I := [a,b]$. Then *f* attains its bounds at least once in [*a, b*]

Proof : If *f* is a constant function, then trivially it attains its bounds at every point of the interval.

Let *f* be a function which is not a constant.

Since *f* is continuous on the closed interval [*a, b*], therefore, it is bounded. Let *m* and *M* be the infimum and supremum of *f*.

We need to showo that there exists point $\alpha, \beta \in$ [*a, b*] such that $f(\alpha) = m,\ f(\beta) = M$.

On contrary, let us suppose *f* does not attain the supremum *M* i.e. $f(x) \neq M,$ for any $x \in [a,b]$.

We consider the function $g(x) = \dfrac{1}{M - f(x)}$ for all $x \in [a,b]$ which is positive for all values of $x \in [a,b]$. Thus the function *g* is continuous and so bounded in [*a, b*]. Let *k* (> 0) be its supremum.

Then,

$$\frac{1}{M - f(x)} \leq k,\ \forall x \in [a,b]$$
$$\Rightarrow f(x) \leq M - \frac{1}{k},\ \forall x \in [a,b]$$

which contradicts the hypothesis that *M* is the supremum of *f* in [*a, b*].

Thus our supposition that *f* does not attain the value *M* leads to a contradiction and therefore *f* attains its supremum for at least one value in [*a, b*].

It may similarly be shown that the function also attains its infimum *m*. Hence, the function attains its bounds at least once in [*a, b*].

Location of roots theorem : Let $f : I \to \mathbb{R}$ be continuous on closed bounded interval *I* : = [*a, b*]. If $f(a) \leq 0 \leq f(b)$ or if $f(a) \geq 0 \geq f(b),$ then there exists a number $c \in (a,b)$ such that $f(c) = 0$

Proof : Let us suppose $f(a) < 0 < f(b)$. We generate a sequence

of intervals by successive bisections. Let $I_1 := [a_1, b_1]$ where $a_1 = a$ and $b_1 = b$. Let $p_1 = (a_1 + b_1)/2$ be the midpoint. If $f(p_1) = 0$, we take $c = p_1$ and we prove the theorem. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$.

If $f(p_1) > 0$, then we set $a_2 = a_1$ and $b_2 = p_1$

If $f(p_1) < 0$, then we set $a_2 = p_1$ and $b_2 = b_1$

In either case, we let $I_2 = [a_2, b_2]$

Thus we have $I_2 \subset I_1$ and $f(a_2) < 0, f(b_2) > 0$.

We continue the bisection process. Suppose that the intervals $I_1, I_2, ..., I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0, f(b_k) > 0$ and we set $p_k = (a_k + b_k)/2$

If $f(p_k) = 0$, we take $c = p_k$ and we prove the theorem. If $f(p_k) \neq 0$, then either $f(p_k) > 0$ or $f(p_k) < 0$.

If $f(p_k) > 0$, then we set $a_{k+1} = a_k$ and $b_{k+1} = p_k$

If $f(p_k) < 0$, then we set $a_{k+1} = p_k$ and $b_{k+1} = b_k$

In either case, we let $I_{k+1} = [a_{k+1}, b_{k+1}]$ such that $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0, f(b_{k+1}) > 0$.

If the process stops by locating a point p_n such that $f(p_n) = 0$, then we are done.

If the process does not stop, then we obtain a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that for every $n \in \mathbb{N}$, we have $f(a_n) < 0$ and $f(b_n) > 0$.

Furthermore, since the intervals are obtained by repeated bisections, the length of I_n is $b_n - a_n = (b-a)/2^{n-1}$. It follows from the Nested interval property that there exists a point $c \in I_n$ for all $n \in \mathbb{N}$. Since $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$, we have $0 \leq c - a_n \leq -a_n = (b-a)/2^{n-1}$ and $0 \leq b_n - c \leq b_n - a_n = b_n - a_n = (b-a)/2^{n-1}$.

Hence it follows that $\lim(a_n) = c = \lim(b_n)$.

Since *f* is continuous at *c*, we have

$$\lim(f(a_n)) = f(c) = \lim(f(b_n))$$

Also since $f(a_n) < 0$ for all $n \in \mathbb{N}$, thus $f(c) = \lim(f(a_n)) \leq 0$.

Also since $f(b_n) \geq 0$ for all $n \in \mathbb{N}$, thus $f(c) = \lim(f(b_n)) \geq 0$.

Thus we conclude that *f*(*c*) = 0 and *c* is a root of *f*.

Intermediate Value Theorem : If *f* is continuous on a closed interval [*a, b*] and *k* is any number between *f*(*a*) and *f*(*b*), then there exists at least a point $c \in [a,b]$ such that $f(c) = k.$

Proof: Suppose that $a < b$ and let $g(x) := f(x) - k.$ Then $g(a) < 0 < g(b).$ By the Location of root theorem, there exists a point *c* with $a < c < b$ such that $0 = g(c) = f(c) - k.$ Thus $f(c) = k.$

If $b < a$ and let $h(x) := k - f(x).$ Then $h(b) < 0 < h(a).$ By the Location of root theorem, there exists a point *c* with $b < c < a$ such that $0 = h(c) = k - f(c).$ Thus $f(c) = k.$

8.7 DEFINITION AND EXAMPLE OF UNIFORM CONTINUITY

Definition: Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function whose domain is *A* and there exists a $\delta > 0$ such that for any numbers $|f(x) - f(u)| < \in$ whenever $|x - u| < \delta.$

Note: From the definition, it is clear that if *f* is uniformly continuous on *A*, then it is continuous at every point in *A*. However, the converse is not true.

Theorem: Non-uniform Continuity Criteria Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}.$ Then the following statements are equivalent:

i. *f* is not uniformly continuous on *A*

ii. There exists an $\in_0 > 0$ such that for every $\delta > 0$ there are points $x_\delta, u_\delta \in A$ such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \in_0$

iii. There exists an $\in_0 > 0$ and two sequences (x_n) and (u_n) in *A* such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \in_0$ for all $n \in \mathbb{N}$

Theorem: A function which is continuous on a closed interval is also uniformly continuous on that interval.

Proof: If *f* is not uniformly continuous on *I*, then by the non-uniform continuity criteria, there exists an $\in_0 > 0$ and two sequences (x_n) and (u_n) in *I* such that $|x_n - u_n| < 1/n$ and $|f(x_n) - f(u_n)| \geq \in_0$ for all $n \in \mathbb{N}.$

Since *I* is bounded, the sequence $\{x_n\}$ is bounded. So there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to an element *z*. Since *I* is

closed, the limit *z* belongs to *I*.

Also the corresponding sub-sequence $\{u_{n_k}\}$ also converges to *z* since

$$|u_{n_k} - z| \le |u_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

Now if *f* is continuous at the point *z*, then both sequences $(f(x_{n_k}))$ and $f(u_{n_k})$ must converge to *f*(*z*). But it is not possible since

$$|f(x_n) - f(u_n)| \ge z_0$$

for all $n \in \mathbb{N}$. Thus our assumption that *f* is not uniformly continuous on *I* implies *f* is not continuous at some point $z \in I$. Therefore, by contra-positivity, *f* is continuous at every point of *I* implies *f* is uniformly continuous on *I*.



8.8 LET US SUM UP

- Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be continuous at *c*, if the following conditions are satisfied:
  1. *f*(*c*) is defined
  2. $\lim_{x \to c} f(x)$ exists
  3. $\lim_{x \to c} f(x) = f(c)$
- Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $B \subseteq A$, then the function *f* is said to be continuous on the set *B* if *f* is continuous at every point of *B*.
- Let $A \subseteq \mathbb{R}$, and $f : A \to \mathbb{R}$ be a function with domain *A* and co-domain $\mathbb{R}$. For any $c \in A$, the function *f* is said to be continuous at *c*, if, given any number $\in > 0$, there exist a number $\delta > 0$ such that for any $x \in A$, $|f(x) - f(c)| < \in$ whenever $0 < |x - c| < \delta$.
- **Jump discontinuity** occurs when the right-hand limit and left-hand limit both exist but are not equal. Hence $\lim_{x \to x} f(x)$ does not exist, violating the second condition.
- **Infinite discontinuity** occurs when one-sided limits (left-hand limit and right hand limit) of the function is infinite. Thus $\lim_{x \to x} f(x)$ does not exist, violating the second condition.
- **Removable discontinuity** occurs when $\lim_{x \to x} f(x)$ exists but is not

equal to $f(c)$. This violates the third condition of definition on continuity.

- Let $A \subseteq \mathbb{R}$ and let f and g be functions from A to $\mathbb{R}$ which are continuous at $x = c$. Then
  (a) $f + g$ continuous at c
  (b) $f - g$ is continuous at c
  (c) fg is continuous at c
  (d) bf is continuous at c
  (e) bg is continuous at c
  (f) f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.
- For $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous functions at $c \in A$ and for $f(A) \subseteq B$, let $g : B \to \mathbb{R}$ be a continuous functions at $b = f(c) \in B$. Then the composition $gof : A \to \mathbb{R}$ is continuous at c.
- For $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous functions on A and for $c \in A$ and $f(A) \subseteq B$, let $g : B \to \mathbb{R}$ be a continuous functions on B. Then the composition $gof : A \to \mathbb{R}$ is continuous on A.
- A function $f : A \to \mathbb{R}$ is continuous at $x = c$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.
- Let $f : I \to \mathbb{R}$ be continuous on closed bounded interval $I := [a, b]$. Then f is bounded on I.
- Let $f : I \to \mathbb{R}$ be continuous function on $I := [a, b]$. Then f attains its bounds at least once in $[a, b]$
- Let $f : I \to \mathbb{R}$ be continuous on closed bounded interval $I := [a, b]$. If $f(a) \leq 0 \leq f(b)$ or if $f(a) \geq 0 \geq f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$
- If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists at least a point $c \in [a, b]$ such that $f(c) = k$.
- Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function whose domain is A and co-domain is $\mathbb{R}$. The function f is said to be uniformly continuous on A if for each $\in > 0$, there exists a $\delta > 0$ such that for any numbers $x, u \in A, |f(x) - f(u)| <$ $\in$ whenever $|x - u| < \delta$.

- If *f* is uniformly continuous on *A*, then it is continuous at every point in *A*. However, the converse is not true.
- Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent :
  - i. *f* is not uniformly continuous on *A*
  - ii. There exists an $\in_0 > 0$ such that for every $\delta > 0$ there are points $x_\delta, u_\delta \in A$ such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta - f(u_\delta)| \geq \in_0$
  - iii. There exists an $\in_0 > 0$ and two sequences (x_n) and (u_n) in *A* such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \in_0$ for all $n \in \mathbb{N}$
- A function which is continuous on a closed interval is also uniformly continuous on that interval.



8.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Check your progress 1

Ans to Q No 1: We have to prove $f(x) = 1/x$ is continuous on $A = \{x \in \mathbb{R} :$ *x* > 0} i.e. we need to prove that for $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \in$ whenever $|x - c| < \delta$

i.e. $\left|\frac{1}{x} - \frac{1}{c}\right| < \in$ whenever $|x - c| < \delta$

But $\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c - x)\right| = \frac{1}{cx}(c - x)$

for $x > 0$

So we need to have an upper bound for the term 1/(*cx*) that holds in some neighbourhood of *c*. Let

$$|x - c| < \frac{1}{c}$$

which implies $\frac{1}{2}c < x < \frac{3}{2}c$

So we have $0 < \frac{1}{cx} < \frac{2}{c^2}$ for $|x - c| < \frac{1}{2}c$

Thus we can write $\left|\frac{1}{x}-\frac{1}{c}\right| \leq \frac{2}{c^2}|x-c|$

Now to make the term on the right hand side less than $\in$, we take

$$|x-c| < \frac{1}{2}c^2 \in$$

Thus choosing $\delta(\in) = \min\left\{2, c^2 \in /2\right\}$, we get

$\left|\frac{1}{x}-\frac{1}{c}\right| < \in$ whenever $|x-c| < \delta(\in)$

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.

Ans to Q No 5: Try yourself.

Answer to Check your progress 2

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.



8.10 FURTHER READING

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7) Bartle, R. G., Sherbert, D. R. (2011). Introduction to real analysis. Hoboken, NJ: Wiley.

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8.11 MODEL QUESTIONS

Q 1: Show that $f(x) = 1/x$ is not uniformly continuous on [0, 1]

Q 2: Show that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty]$

Q 3: Show that the function $f(x) = x^2$ is uniformly continuous on [–1, 1]

Q 4: Prove that $f(x) = \dfrac{x^2 - 4}{x^2 + 1}$ is continuous at $x = 2$

Q 5: Show that the function $f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$

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